We review some novel methods of IBP reduction, based on algebraic geometry and non-commutative algebra.

I. IBP WITHOUT DOUBLE PROPAGATORS

The idea [1] is the following: usually the hard problem of an IBP reduction is the reduction of integral with high numerator powers, namely the integrals with large negative indices. In order to reduce these integrals, even if we start from integrals without propagators, in the intermediate steps we get a lot of integrals with double propagators. The Gaussian elimination must take care of these integrals and extra steps of cancellation happen. Is there a way to avoid integrals with double propagators?

The answer to carefully pick up the vector $v\mu$ such that double propagator integrals do not appear in the beginning. Such a choice, cannot be found by linear algebra, but need to be done by the algebraic geometry tool: *syzygy*.

Suppose that we only focus on integrals without doubled propagator, then we would like to start with an IBP such that,

$$0 = \int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^{L} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1 \ldots D_k} \tag{1}$$

and assume v may not be the simple momenta like $p_1, \ldots p_E$ and $l_1, \ldots l_n$, but a combination of them with *polynomial coefficients in the loop momenta.* In this formalism, we start with integrals without double propagator.

But the derivative would introduce double propagators. Note that

$$\frac{\partial}{\partial l_i^\mu} \frac{1}{D_j} = -\frac{\partial D_j}{\partial l_i^\mu} \frac{1}{D_j^2} \tag{2}$$

If we require that [2],

$$\sum_{i=1}^{L} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} = f_j D_j \tag{3}$$

then the double propagator is gone. Here f_j is a polynomial coefficients in the loop momenta.

The equation (3) is a “syzygy” equation. Although it looks like a linear equation in v_i and f_j, the requirement that both v_i and f_j are polynomials makes it impossible to be solved in the framework of linear algebra. This is again an algebraic geometry problem. Syzygy means to solve homogenous linear equations with polynomial solution only.

We take a solid example. Consider the one-loop massless triangle diagram with

$$D_1 = l^2, \quad D_2 = (l - p_1)^2, \quad D_3 = (l - p_1 - p_2)^2, \quad D_4 = (l + p_4)^2 \tag{4}$$

Here D_4 is an ISP. We want to find IBP for integrals $G[n_1, n_2, n_3, n_4]$ with $n_1 \leq 1$, $n_2 \leq 1$ and $n_3 \leq 1$ (no double propagator for the three propagators).

We can set

$$v = a_1 p_1 + a_2 p_2 + a_3 p_4 + a_4 l \tag{5}$$

where a's are polynomials of the loop momenta l. With a simple calculation of the derivatives, we arrive at such a syzygy equation.

$$\begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 2y & -y & 0 & 0 \\ 2x_1 & 2x_2 - s & 2x_3 - t & 2y - 2x_1 & 0 & 2x_1 - y & 0 \\ 2x_1 - s & 2x_2 - s & s + 2x_3 & 2y - 2x_1 - 2x_2 & 0 & 0 & -s - y + 2x_1 + 2x_2 \end{pmatrix} \mathbf{u} = \mathbf{0} \tag{6}$$

where $\mathbf{u} = \{a_1, a_2, a_3, a_4, f_1, f_2, f_3\}$.

Any syzygy equation can be solved by Schreyer algorithm, based on Groebner basis computations [3]. This algorithm is implemented in the computational algebraic geometry software sc Singular. Using Singular, we get the solutions,

$$\mathbf{u}^{(1)} = \Big(-2x_2, 2x_1 - 2y, 0, 4x_2 - s, 4x_2 - 2s, 4x_2, 4x_2 \Big) \tag{7}$$

$$\mathbf{u}^{(2)} = \Big(-2x_3, y, 2x_1 - y, s - 2x_2 + 2x_3, 2s - 2x_2 + 2x_3, s + t - 2x_2 + 2x_3, 2x_3 - 2x_2 \Big) \tag{8}$$

$$\mathbf{u}^{(3)} = \Big(-y, -y, 0, -s + 2x_1 + 2x_2, -2s + 2x_1 + 2x_2, -s + 2x_1 + 2x_2, 2x_1 + 2x_2 \Big) \tag{9}$$

$$\mathbf{u}^{(4)} = \Big(y(s+t), ty - 2x_3 y, 2x_2 y - sy, y(-s-t), \tag{10}$$

$$x_1(2s + 2t) + y(-2s - 2t) - 2sx_3 + 2tx_2, y(-2s - 2t), y(-2s - 2t) \Big) \tag{11}$$

Consider the first solution, then we get

$$v = -2x_2 p_1 + (-2y + 2x_1) p_2 + (-s + 4x_2) l \tag{12}$$

Put it to the IBP equation, immediately we get the IBP relation

$$(D - 4)sG[1, 1, 1, 0] + (2D - 6)G[1, 0, 1, 0] + (2D - 6)G[1, 1, 0, 0] = 0 \tag{13}$$

So there is no double propagator. We see that it is a simple syzygy relation which simply reduce

$$G[1, 1, 1, 0] = -\frac{2D - 6}{s(D - 4)} G[1, 0, 1, 0] \tag{14}$$

The normal IBP process has to do Gaussian eliminations to reduce integrals with double propagators to get the result above.

Note that for any $\mathbf{u}^{(i)}$ above, we can multiply it by a polynomial in loop momenta, then we still get IBPs without double propagators (with higher numerator power). Mathematically, the solution set is a *module* (vector version of an ideal).

We remark that this method also works for this situation when the input integrals have double propagators. In this case, the method avoids triple propagators and extra double propagators.

For research purpose, it is important not to use the momentum integral but the Baikov integral representation [4, 5]. In the Baikov representation via module intersection method, it is much easier to solve the syzygy equation, simplify the scalar products and impose unitarity cuts to divide the IBP reduction problem. Several IBP reduction examples, which cannot be done with the popular softwares, can be solved in this way.

II. IBP WITHOUT ISPS

Sometime we have the opposite request: we want to reduce integrals with double propagator (or multiple propagators) without considering ISPs in the numerator. This can be done with Lee-Pomeransky representation and syzygy techniques [6].

Note that for Lee-Pomeransky representation, naturally we do not need the ISPs, unlike the Baikov representation. We may want to start with the total derivative in Lee-Pomeransky representation,

$$\int_0^\infty dz_1 \ldots dz_k \sum_{i=1}^k \partial_i (Q_i G^{-D/2}) \tag{15}$$

Here Q_i's are polynomials of z_i's. Note that the first problem is that the total derivative is usually not integrated to zero, because of the surface term at $z_i = 0$. We can only state that

$$\int_0^\infty dz_1 \ldots dz_k \sum_{i=1}^k \partial_i (Q_i G^{-D/2}) = \text{lower sector integrals} \tag{16}$$

The second problem is that

$$\partial_i (G^{-D/2}) = -\frac{D}{2} \frac{\partial G}{\partial z_i} G^{-D/2-1} \tag{17}$$

So we get an IBP with the mixture integrals in the D dimension and $D-2$ dimension. Such an IBP is polluted by dimension shift identities. In order to avoid this, again, we put a constraint on the polynomial Q_i's,

$$\left(\sum_{i=1}^{k} Q_i \partial_i G\right) + fG = 0 \tag{18}$$

where f is a polynomial of z's. If (18) is satisfied, then the $D-2$ dimensional integrals disappear.

Again (18) is not a linear equation, since we require that both Q_i's and f to be polynomials. It is a syzygy equation of polynomials. We can again use *Singular* to solve it.

After getting the syzygy solutions, although it is possible to plug it in the (18) to get the IBP relation, we have to by hand recover the surface terms and convert them to lower sector integrals. Instead, Lee provides an operator language which can deal with the surface term automatically [6].

Consider $G[n_1, \ldots n_k]$ as a function on the lattice $\mathbb{Z}^k$ to $\mathbb{C}$. Define two operators A_i and B_i which map such a lattice function to another lattice function.

$$(\hat{A}_i G)(n_1, \ldots, n_k) = n_i G(n_1, ..., n_i + 1, \ldots, n_k), \tag{19}$$

$$(\hat{B}_i G)(n_1, \ldots, n_k) = G(n_1, ..., n_i - 1, \ldots, n_k), \tag{20}$$

Note that $\hat{A}_i, \hat{B}_i$'s are not maps on the lattice, but maps on the lattice functions. The commutator relation is,

$$[\hat{A}_i, \hat{B}_j] = \delta_{ij} \, . \tag{21}$$

So they are ladder operators. In the operator language the IBP relation from syzygies read

$$\left(\sum_{i=1}^{k} Q_i(\hat{A}_1, \ldots \hat{A}_k)\hat{B}_i + \frac{D}{2} Q[\hat{A}_1, \ldots \hat{A}_k]\right) G[n_1 \ldots n_k] = 0 \tag{22}$$

In this formalism, the surface terms are automatically included.

This method may be very useful for the future IBP algorithms.

III. OPERATOR IBP

It is possible to formally rewrite all IBP relations, the traditional ones, the ones without double propagators, the ones without ISP to the operator form [7]:

$$(\text{Operator}) G[n_1, \ldots n_k] = 0 \tag{23}$$

for arbitrary indices $n_1, \ldots n_k$. Such an operators form a left ideal of a non-commutative algebra. Note that since the ladder operators do not commute, we cannot consider the polynomial ring or ideals. Instead we consider the non-commutative algebra and the left ideal.

Normal linear algebra IBP reduction, or the novel IBP reduction without propagator or ISPs, all involves building a tower of linear relations and then do the Gaussian elimination. However, in noncommutative algebra, we may consider the Groebner basis of the left ideal of IBPs. Then a division toward the left ideal seems to give a direct reduction, without "diagonalizing" the IBP system [7].

However, it is much more complicated to compute a Groebner basis for a non-commutative algebra than that for a polynomial ring or module. So far, this direction is not practical yet.

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